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Rigidity about two linear structures on the space of measured laminations*

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Abstract: By modeling the space of projective measured laminations in the cotangent space to Teichmüller space via hyperbolic length functions and extremal length functions, we associate two classes of linear structures to the space of measured laminations. We prove that both of these two linear structures are rigid: the induced linear structures on different Riemann surfaces are different.

Key words: Teichmüller space; measured laminations; linear structures; hyperbolic length; extremal length

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关于可测叶状结构空间上两种线性结构的刚性

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摘　要：通过黎曼曲面的双曲长度函数与极值长度函数，将可测叶状结构空间实现为Teichmüller空间的余切空间，从而诱导了可测叶状结构空间上的两种线性结构。证明了这两种线性结构都具有刚性性质，也即不同的黎曼曲面诱导不同的线性结构。

关键词：Teichmüller空间；可测叶状结构；线性结构；双曲长度；极值长度

Royden[1] proved the isometry rigidity of the Teichmüller metric in the sense that every isometry is induced by a mapping class of the underlying surface. Since then, the rigidity phenomenon in Teichmüller theory has been one of the main research interests in the field. So far, we know that the Teichmüller metric[1-3], the Weil-Petersson metric[4-5], the Thurston metric[6-8], the mapping class group[9-10], the curve complex and its relatives[11-12], are all isometrically rigid. Ivanov proposed the following metaconjecture:

Ivanov Metaconjecture: Every object naturally associated to a surface and having a sufficiently rich structure has the extended mapping class group as its group of automorphisms. Moreover, this can be proved by a reduction to the theorem about the automorphisms of the curve complex.

In this paper, we consider a rigidity problem about the linear structures on the space of measured laminations, adding one more item to the rigidity list.

Let S be an orientable surface of genus g with n punctures such that the Euler characteristic $2-2g-n<0$. Let X be a hyperbolic metric on S. A geodesic lamination on X is a closed subset which can be foliated by simple

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geodesics. A transverse invariant measure μ on a geodesic lamination L is an assignment of a non-negative number $\mu(k)$ to each arc k which is transverse to L and whose endpoints are disjoint from L, such that

- **countable additivity**: for any countably subdivision $\{k_i\}_{i=1}^{\infty}$ of k with $\partial k_i \cap L = \varnothing$, we have $\mu(k) = \sum_{i=1}^{\infty} \mu(k_i)$;
- **transverse invariance**: for any arc k' obtained from k by an isotopy respecting L, we have $\mu(k) = \mu(k')$;
- **supported on** L: for all arcs disjoint from L, we have $\mu(k) = 0$.

A measured geodesic lamination is a geodesic lamination together with a transverse invariant measure. The simplest examples are the isotopy classes of simple closed geodesics on X. More precisely, for a simple closed curve α and a transverse arc k, $\alpha(k)$ is defined to be the cardinality of $\alpha \cap k$. Let C be the set of isotopy classes of simple closed curves on S. Each measured geodesic lamination μ defines a function μ: $C \to \mathbf{R}$, which associates to $\alpha \in C$ the geometric intersection number $i(\mu, \alpha) := \inf \mu(\alpha)$, where the infimum is taken over all representatives in α. Let $ML(X)$ be the space of measured geodesic laminations on X equipped with the weak topology from $\mathbf{R}^C$, the infinite product of $\mathbf{R}$ indexed by C. With respect to this topology, the set of weighted simple closed curves $\mathbf{R}_{>0} \times C$ is dense in $ML(X)$. The intersection pairing i: $ML(X) \times C \to \mathbf{R}$ extends continuously to a homogeneous function on $ML(X) \times ML(X)$. By the one-to-one correspondence between simple closed geodesics of X and simple closed geodesics of any other hyperbolic metric on S, the definition above does not depend on the choice of X. For this reason, we shall denote by $ML(S)$ the space of measured geodesic laminations on S without referring to any specific choice of hyperbolic metric. Let $PML(S) := ML(S)/\mathbf{R}^+$ be the space of projective equivalence classes of measured geodesic laminations. (For more details about measured geodesic laminations, we refer to [13-14].)

It is known that $ML(S)$ admits a piecewise linear structure[14]. This piecewise linear structure is very important in Teichmüller theory. In this note, we shall discuss some other linear structures on $ML(S)$ by modelling it into the tangent space (resp. cotangent space) of the Teichmüller space $T(S)$ via hyperbolic length functions (resp. extremal length functions), see Proposition 1 and Proposition 4. Pan[8] proved an analogue of Royden's theorem in the setting of the Thurston metric. A key step in his proof is a related linearity rigidity of linear structures on $ML(S)$.

1　Modeled in the cotangent space via hyperbolic length functions

1.1　Teichmüller space of hyperbolic metrics

A marked hyperbolic surface is a pair (X, f) where X is a hyperbolic surface and f: $S \to X$ is an orientation-preserving homeomorphism. Two marked hyperbolic surfaces (X, f) and (X', f') are said to be equivalent if and only if $f' \circ f^{-1}$: $X \to X'$ is homotopic to an isometry. The Teichmüller space $T(S)$ is defined to be the set of equivalence classes of marked hyperbolic surfaces. In the following, we shall simply denote the equivalence class $[X, f]$ by X. It is well known that $T(S)$ is homeomorphic to $\mathbf{R}^{-3(2-2g-n)}$.

1.2　Hyperbolic length functions

Recall that C is the set of isotopy classes of simple closed curves on S. For each $\alpha \in C$ and $X \in T(S)$, there is a unique geodesic representative of α on the hyperbolic surface X. Let $l_\alpha(X)$ be the length of this geodesic representative. This defines a function on $T(S) \times C$:

$$l: T(S) \times C \to \mathbf{R},$$
$$(X, \alpha) \mapsto l_\alpha(X),$$

which admits a unique continuous extension:

$$l: T(S) \times ML(S) \to \mathbf{R},$$

such that $l_{t\mu}(X) = t l_\mu(X)$ for all $t > 0$, $\mu \in ML(S)$ and $X \in T(S)$. For each measured geodesic lamination μ, l_μ:

$T(S)\to\mathbf{R}$ is real analytic (for instance see [15, Corollary 2.2]).

1.3 Linear structures via hyperbolic length functions

Proposition 1 [16, Theorem 5.1] For any hyperbolic surface X, the map

$$\begin{aligned} PML(S) &\to T_X^*T(S),\\ [\mu] &\mapsto d_X \log l_\mu, \end{aligned}$$

embeds $PML(S)$ as the boundary of a convex neighborhood of the origin.

By identifying $PML(S)$ with the subset $\{\lambda\in ML(S): l_\lambda = 1\}$, we obtain a homeomorphism:

$$\begin{aligned} DL_X: ML(S)\cup\{0\} &\to T_X^*T(S),\\ \mu &\mapsto d_X l_\mu. \end{aligned}$$

In this way, we associate to $ML(S)\cup\{0\}$ a linear structure induced from the linear structure on $T_X^*T(S)$.

To see the dependence of the linear structures on the underlying hyperbolic surfaces, we consider the map $DL_Y\circ DL_X^{-1}$: $T_X^*T(S)\to T_Y^*T(S)$ which is the composition of DL_X^{-1}: $T_X^*T(S)\to ML(S)\cup\{0\}$ and DL_Y: $ML(S)\cup\{0\}\to T_Y^*T(S)$. In particular,

$$DL_Y\circ DL_X^{-1}\left(d_X l_\mu\right) = d_Y l_\mu,\ \ \forall\mu\in ML(S).$$

Theorem 1 $DL_Y\circ DL_X^{-1}$: $T_X^*T(S)\to T_Y^*T(S)$ is linear if and only if $Y = X$.

The primary tool we use to prove Theorem 1 is the earthquake deformation.

1.4 Earthquakes

Given any hyperbolic surface $X\in T(S)$ and $(t,\alpha)\in\mathbf{R}\times C$, the cutting and gluing operation defines a new hyperbolic surface $E_\alpha^t(X)$ by cutting X along α and gluing it back with a left twist of (hyperbolic) length t ($t<0$ means a right twist). One can extend this operation to the set of weighted simple closed curves by setting $E_{a\alpha}^t(X) := E_\alpha^{at}(X)$ for any $a>0$, $t\in\mathbf{R}$ and $\alpha\in C$. In [17], Kerckhoff showed that the cutting and gluing operation can also be extended to any measured geodesic lamination.

Proposition 2[17] Suppose $\mu\in ML(S)$, then for any sequence of weighted simple closed curves $a_i\alpha_i$ converging to μ, the limit $\lim\limits_{i\to\infty} E_{a_i\alpha_i}^t$ exists. Moreover, the limit is independent of the choice of the approximating sequence $a_i\alpha_i$.

The limit $E_\mu^t(X) := \lim\limits_{i\to\infty} E_{a_i\alpha_i}^t$ is said to be the time t left earthquake of X along μ. The image $\{E_\mu^t(X)\}_{t\in\mathrm{R}}$ is called an earthquake line directed by μ.

Proposition 3[17]

(i) For any two distinct hyperbolic surfaces $X, Y\in T(S)$, there exists a unique earthquake path passing through them.

(ii) For any $X\in T(S)$, and any two measured geodesic laminations λ and μ, the length functions l_λ: $T(S)\to\mathbf{R}$ is convex along the earthquake line $\{E_\mu^t(X)\}_{t\in\mathrm{R}}$. Moreover, if $i(\lambda,\mu)>0$, then l_λ is strictly convex along $\{E_\mu^t(X)\}_{t\in\mathrm{R}}$.

1.5 Proof of Theorem 1

Let $\Pi_{XY} := DL_Y\circ DL_X^{-1}$. If $Y = X$, then Π_{XY} is the identity map, which is linear. We now consider the converse. In the following, we assume that Π_{XY} is linear.

Suppose to the contrary that $Y\neq X$. By Proposition 3, there exists a unique measured lamination δ such that $E_\delta^1(X) = Y$. Let λ be a measured foliation which intersects δ. Then there exists a unique $\mu\in ML(S)$ such that

$$d_X l_\lambda + d_X l_\mu = d_X l_\delta \in T_X^*T(S). \tag{1}$$

By assumption, Π_{XY} is linear, then

$$d_Y l_\lambda + d_Y l_\mu = d_Y l_\delta \in T_Y^*T(S). \tag{2}$$

Since λ intersects δ, it follows from Proposition 3 that l_λ is strictly convex along the earthquake line $\{E_\delta^t(X)\}_{t\in\mathrm{R}}$. Combing with that l_δ is a constant on the earthquake line $\{E_\delta^t(X)\}_{t\in\mathrm{R}}$, we see that l_μ is not a constant on the earthquake line $\{E_\delta^t(X)\}_{t\in\mathbb{R}}$. Therefore, μ intersects δ. This in turn implies that both l_λ and l_μ are strictly convex along the earthquake line $\{E_\delta^t(X)\}_{t\in\mathbb{R}}$. Let $\dot{E}_\delta$ be the tangent vector field along the earthquake line $\{E_\delta^t(X)\}_{t\in\mathbb{R}}$. Then

$$\dot{E}_\delta(X)(l_\lambda) < \dot{E}_\delta(Y)(l_\lambda),$$
$$\dot{E}_\delta(X)(l_\mu) < \dot{E}_\delta(Y)(l_\mu).$$

In particular,

$$\dot{E}_\delta(X)(l_\lambda + l_\mu) < \dot{E}_\delta(Y)(l_\lambda + l_\mu).$$

On the other hand, it follows from (1) and (2) that

$$\dot{E}_\delta(X)(l_\lambda + l_\mu) = \dot{E}_\delta(X)(l_\delta) = 0, \qquad \dot{E}_\delta(Y)(l_\lambda + l_\mu) = \dot{E}_\delta(Y)(l_\delta) = 0.$$

Contradiction!

Remark 1 If λ and μ are disjoint measured laminations, then $\lambda + \mu$ is also a measured lamination. In this case, $d_X l_{\lambda+\mu} = d_X l_\lambda + d_X l_\mu$ holds at each point $X \in T(S)$. In particular, for each point $Y \in T(S)$, we have

$$DL_Y \circ L_X^{-1}(d_X l_\lambda + d_X l_\mu) = DL_Y \circ DL_X^{-1}(d_X l_{\lambda+\mu}) = d_Y l_\lambda + d_Y l_\mu.$$

We guess that the converse is also true, namely, $DL_Y \circ DL_X^{-1}(d_X l_\lambda + d_X l_\mu) = d_Y l_\lambda + d_Y l_\mu$ holds only when λ and μ are disjoint.

2 Modeled in the cotangent space via Extremal length functions

2.1 Teichmüller space of conformal structures

A marked conformal structure on S is a pair (X, f) where X is a Riemann surface and $f: S \to X$ is an orientation-preserving homeomorphism. Two marked conformal structures (X, f) and (X', f') are said to be *equivalent* if and only if $f' \circ f^{-1}: X \to X'$ is homotopic to a conformal map. The Teichmüller space $T(S)$ can also be defined to be the set of equivalence classes of marked conformal structures. In the following, we shall simply denote the equivalence class $[X, f]$ by X.

2.2 Extremal length functions

Let α be an isotopy class of simple closed curve, and X be a (marked) Riemann surface. A conformal metric on X is a metric which can be expressed as $\rho(z)|dz|$ locally. The extremal length of α on X is defined by:

$$Ext_\alpha(X) := \sup_\rho \frac{l_\rho^2(\alpha)}{\text{Area}(\rho)},$$

where the sup ranges over all the conformal metrics on X, $\text{Area}(\rho)$ is the area of X endowed with the metric ρ, and $l_\rho(\alpha) := \inf \int_{\alpha'} \rho|dz|$, where the infimum is taken over all representatives α' in α. It is clear that

$$\frac{l_{a\rho}^2(\alpha)}{\text{Area}(a\rho)} = \frac{l_\rho^2(\alpha)}{\text{Area}(\rho)},$$

for any positive constant a. The extremal length is a conformal invariant. Kerckhoff[18, Proposition 3] extended the definition of extremal length from weighted simple closed curves to measured laminations:

$$Ext: T(S) \times ML(S) \to \mathbf{R},$$

such that $Ext_{t\mu}(X) = t^2 Ext_\mu(X)$ for all $t > 0$, $\mu \in ML(S)$ and $X \in T(S)$. Moreover, for each measured lamination λ, the extremal length function $Ext_\lambda: T(S) \to \mathbf{R}$ is differentiable and positive.

2.3 Linear structures via extremal length functions

Proposition 4[19, Theorem 4.1] For any complex structure X, the map

$$PML(S) \to T_X^* T(S),$$
$$[\mu] \mapsto d_X \log \sqrt{Ext_\mu},$$

embeds $PML(S)$ as the boundary of a convex neighborhood of the origin.

By identifying $PML(S)$ with the subset $\{\lambda \in ML(S) : Ext_\lambda(X) = 1\}$, we obtain a homeomorphism:

$$DE_X: ML(S) \cup \{0\} \to T_X^*T(S),$$
$$\mu \mapsto d_X\sqrt{Ext_\mu}.$$

In this way, we associate to $ML(S) \cup \{0\}$ another linear structure induced from the linear structure on $T_X^*T(S)$. We consider the map

$$DE_Y \circ DE_X^{-1}: T_X^*T(S) \to T_Y^*T(S),$$

which is the composition of $DE_X^{-1}: T_X^*T(S) \to ML(S)$ and $DE_Y: ML(S) \to T_Y^*T(S)$. In particular,

$$DE_Y \circ DE_X^{-1}\left(d_X\sqrt{Ext_\mu}\right) = d_Y\sqrt{Ext_\mu}.$$

Theorem 2 $DE_X \circ DE_Y^{-1}$ is linear if and only if $Y = X$.

Proof For each $\mu \in ML(S)$, there exists a unique holomorphic quadratic differential $Q(\mu, X)$ on X such that for every isotopy class α of simple closed curve, the geometric intersection number between α and the horizontal measured lamination of $Q(\mu, X)$ is the same as the intersection number between α and μ. Let $\mu_X^\perp$ be the vertical measured lamination of $Q(\mu, X)$. By Gardiner's formula[20, Theorem 8], (see also [21, Theorem 1.2]), we see that

$$d_X\sqrt{Ext_\mu} = -d_X\sqrt{Ext_{\mu_X^\perp}} \in T_X^*T(S).$$

Similarly, we have

$$d_Y\sqrt{Ext_\mu} = -d_Y\sqrt{Ext_{\mu_Y^\perp}} \in T_Y^*T(S).$$

Since $DE_X \circ DE_Y^{-1}: T_Y^*T(S) \to T_X^*T(S)$ is linear, it follows that

$$d_Y\sqrt{Ext_\mu} = -d_Y\sqrt{Ext_{\mu_{\mu_Y^\perp}}} = -DE_Y \circ DE_X^{-1}(d_X\sqrt{Ext_\mu})$$
$$= DE_Y \circ DE_X^{-1}(-d_X\sqrt{Ext_\mu}) = DE_Y \circ DE_X^{-1}(d_X\sqrt{Ext_{\mu_X^\perp}})$$
$$= d_Y\sqrt{Ext_{\mu_X^\perp}}.$$

Therefore, $\mu_X^\perp = \mu_Y^\perp$. Consequently, $X = Y$.

Remark 2 Using the theory of lines of minima[22], we can prove Theorem 1 by the method above. More precisely, let X be a hyperbolic surface and λ a maximal lamination on X, i.e. λ intersects every simple closed curve on X. Let μ be the unique measured lamination on X such that $d_Xl_\lambda + d_Xl_\mu = 0$. By Theorem 3.4 of [22], for such a pair (λ, μ), there exists a unique hyperbolic surface X in the Teichmüller space $T(S)$ such that $d_Xl_\lambda + d_Xl_\mu = 0$. This implies that $DL_Y \circ DL_X^{-1}: T_X^*T(S) \to T_Y^*T(S)$ is linear if and only if $Y = X$, which is exactly Theorem 1.

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